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SEMIANALYTIC DISTRIBUTIONS
IN FOUR FERMION NEUTRAL CURRENT PROCESSES [*]

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ABSTRACT

Analytical formulae for triple differential distributions $\mathrm{d}^3\sigma/(\mathrm{d}\cos\theta \mathrm{d}s_1 \mathrm{d}s_2)$ in the neutral current process $e^+e^- \to f_1\bar{f}_1 f_2\bar{f}_2$ are given. They allow to obtain angular distributions, rapidity distributions and transversal momentum distributions of one fermion pair by only two numerical integrations. Cuts can be applied to the integration variables.

1. Introduction

Many events of four fermion final states are expected to be observed at LEPII and NLC (next linear collider). The accuracy of the theoretical description of such processes should match the future experimental precision. As is known from LEPI, both, semianalytical programs and Monte Carlo generators are needed to understand the physics of experimental data.

Semianalytical calculations lead to fast `FORTRAN` codes with high numerical precision. Although a semianalytical calculation will never be able to simulate a real detector, it should be as differential as reasonable. This allows to predict observables closer to the experiment and to make more meaningful comparisons with other groups [1,2].

The semianalytic approach has been applied earlier to the calculation of four fermion final states of neutral current process [3] $e^+e^- \to f_1\bar{f}_1 f_2\bar{f}_2$ and Higgs production in the same reaction [4]. There, analytical formulae for the *double* differential cross section were given. The total cross section was obtained by two numerical integrations. Differential distributions over the invariant energy of one fermion pair $(f_1, \bar{f}_1)$ were obtained by one integration.

In this contribution, we present analytical formulae for *triple* differential cross

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sections. In addition to the invariant energies of the two fermion pairs, we keep the cosine of the angle between the 3-vector of the momentum of one fermion pair and the beam axis as an additional parameter. The resulting analytical formulae turn out to be not much longer than the double differential ones. However, the formulae presented here may be used to get new distributions by two numerical integrations: angular distributions , rapidity distributions and transversal momentum distributions of one fermion pair. These distributions can be obtained including additional cuts on the remaining integration variables.

We start with the process-independent description of the phase space in section 2 and give formulae for the cross section and distributions in section 3. Section 4 contains a brief summary. Detailed formulae for the kinematical functions of the triple differential distributions are given in the appendix. Numerical applications will be presented elsewhere.

2. Phase space

We are interested in the cross section and in distributions of the reaction

$$e^+(k_1)e^-(k_2) \rightarrow f_1(p_1)\bar{f}_1(p_2)f_2(p_3)\bar{f}_2(p_4) \tag{1}$$

with $p_i^2 = m_i^2, \quad i = 1, \ldots, 4$.

The eightdimensional phase space of the four particle final state is parametrized as

$$
\begin{aligned}
d\Gamma &= \prod_{i=1}^{4} \frac{d^3 p_i}{2p_i^0} \times \delta^4(k_1 + k_2 - \sum_{i=1}^{4} p_i) \\
&= 2\pi \frac{\sqrt{\lambda(s, s_1, s_2)}}{8s} \frac{\sqrt{\lambda(s_1, m_1^2, m_2^2)}}{8s_1} \frac{\sqrt{\lambda(s_2, m_3^2, m_4^2)}}{8s_2} ds_1 ds_2 d\Omega d\Omega_1 d\Omega_2. \tag{2}
\end{aligned}
$$

λ is the kinematical function,

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \quad \lambda \equiv \lambda(s, s_1, s_2), \tag{3}$$

and the invariants s, s_1 and s_2 are

$$s = (k_1 + k_2)^2, \quad s_1 = (p_1 + p_2)^2, \quad s_2 = (p_3 + p_4)^2. \tag{4}$$

We integrate over every spherical angle Ω (Ω_1, Ω_2) of the e^+e^- pair ($f_1\bar{f}_1$, $f_2\bar{f}_2$ pair) in its rest frame. The spherical angles are decomposed in two integrations, $d\Omega_i = d\cos\theta_i d\phi_i$ with the kinematical ranges

$$-1 \le \cos\theta_i \le 1, \quad 0 \le \phi_i \le 2\pi. \tag{5}$$

In particular, we have $d\Omega = d\cos\theta d\phi$, where $\cos\theta = c$ is the cosine of the angle between the 3-vectors $(\vec{p}_1 + \vec{p}_2)$ and $\vec{k}_1$. In the case without transversal beam polarization, the integration over the rotation angle ϕ around the beam axis is trivial, giving 2π.

In the following, we will integrate the squared matrix element over the four remaining angular variables $\cos\theta_1, \phi_1, \cos\theta_2$ and ϕ_2. This results to analytic functions depending on the remaining variables c, s_1 and s_2. They enable us to calculate not only total cross sections but also angular distributions $d\sigma/dc$, transversal momentum distributions $d\sigma/dp_{iT}$ or rapidity distributions $d\sigma/dy_i$ of the "compound particle" $(f_i, \bar{f}_i, \ i = 1,2)$. These new distributions are useful for comparisons with Monte Carlo programs and for more reliable experimental predictions.

The transformation of the phase space volume to the new integration variables is

$$\int_{-1}^{1} dc \int_{s_1^-}^{s_1^+} ds_1 \int_{s_2^-(s_1)}^{s_2^+(s_1)} ds_2 = 2\sqrt{s} \int_{y_1^-}^{y_1^+} dy \int_{p_{1T}^-(y_1)}^{p_{1T}^+(y_1)} dp_{1T} \int_{s_2^-(p_{1T},y_1)}^{s_2^+(p_{1T},y_1)} ds_2$$

$$= 2\sqrt{s} \int_{p_{1T}^-}^{p_{1T}^+} dp_{1T} \int_{y_1^-(p_1 T)}^{y_1^+(p_{1T})} dy_1 \int_{s_2^-(p_{1T},y_1)}^{s_2^+(p_{1T},y_1)} ds_2 \qquad (6)$$

with similar formulae for p_{2T} and y_2. The old variables c and s_1 have to be expressed through the new variables p_{1T} and y:

$$c = \sqrt{\frac{1 + p_{1T}^2/(m_1+m_2)^2}{1 + \coth^2(y)\, p_{1T}^2/(m_1+m_2)^2}},$$

$$s_1 = 2\sqrt{s}\sqrt{(m_1+m_2)^2 + p_{1T}^2}\, \cosh y + s_2 - s \qquad (7)$$

The kinematical ranges of the integration variables s_1 and $s_2(s_1)$ are

$$\begin{aligned} s_1^- &= (m_1+m_2)^2, & s_1^+ &= (\sqrt{s} - m_3 - m_4)^2, \\ s_2^-(s_1) &= (m_3+m_4)^2, & s_2^+(s_1) &= (\sqrt{s} - \sqrt{s_1})^2. \end{aligned} \qquad (8)$$

For simplicity, we give the other kinematical borders only for the case of massless final fermions:

$$\begin{aligned} y_1^- &= -\infty, & y_1^+ &= \infty, \\ p_{1T}^-(y_1) &= 0, & p_{1T}^+(y_1) &= \sqrt{s}/\cosh y, \\ p_{1T}^- &= 0, & p_{1T}^+ &= \sqrt{s}, \\ y_1^-(p_{1T}) &= -\text{arcosh}\frac{\sqrt{s}}{p_{1T}}, & y_1^+(p_{1T}) &= \text{arcosh}\frac{\sqrt{s}}{p_{1T}} \end{aligned} \qquad (9)$$

The borders of $s_2(y_1, p_{1T})$ are

$$s_2^-(y_1, p_{1T}) = \max\left\{ 0, s - 2\sqrt{s}\, p_{1T}\cosh y \right\}, \quad s_2^+(y_1, p_{1T}) = (\sqrt{s} - p_{1T}\cosh y)^2. \quad (10)$$
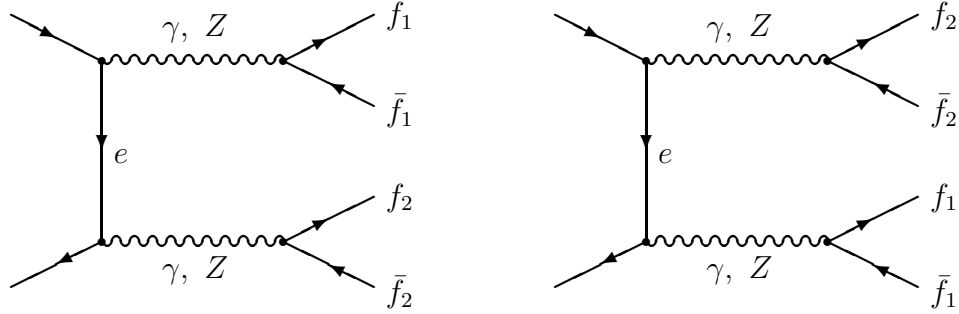
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Fig. 1: *The* `crab` *diagrams.*

3. Cross sections and distributions

We present the formulae for the triple differential distribution of the reaction (1), where only neutral gauge bosons are exchanged. The initial electron-positron pair and the final fermions f_1 and f_2 are all different and none of them belongs to one electroweak multiplet. The reaction (1) is then described by 24 Feynman diagrams which exchange photons and Z-bosons. In the case of four quarks in the final state, there are 8 additional diagrams with gluon exchange. One can divide these diagrams in three different types. The first type which we call `crab` (these diagrams are also called conversion diagrams) is shown in fig. 1. These diagrams can be considered as $e^+e^- \to f_1\bar{f}_1$ with initial state radiation of a $f_2\bar{f}_2$-pair (or $e^+e^- \to f_2\bar{f}_2$ with initial state radiation of a $f_1\bar{f}_1$-pair). The second type which we call `deers` (they are also called annihilation diagrams) can be considered as $e^+e^- \to f_1\bar{f}_1$ with radiation of a $f_2\bar{f}_2$-pair from the final state (f_1-`deer`), see fig. 2. A different set of diagrams is obtained for $e^+e^- \to f_2\bar{f}_2$ with radiation of a $f_1\bar{f}_1$-pair from the final state (f_2-`deer`). The process is symmetric in f_1 and f_2.

The squared matrix element of the considered reaction is calculated using the symbolic manipulation program `FORM` [5]. As a result, analytical formulae for the triple differential cross section are obtained,

$$\frac{\mathrm{d}^3\sigma(c, s, s_1, s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2} = \sum_{k=1}^{6} \frac{\mathrm{d}^3\sigma_k(c, s, s_1, s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2}. \tag{11}$$

The summation runs over the squares and interferences between the three different types of diagrams: $k = 1$ corresponds to the square of `crabs`, and $k = 2, 3$ to that of f_1-`deers` and f_2-`deers`. Further, $k = 4, 5$ and 6 correspond to the interferences
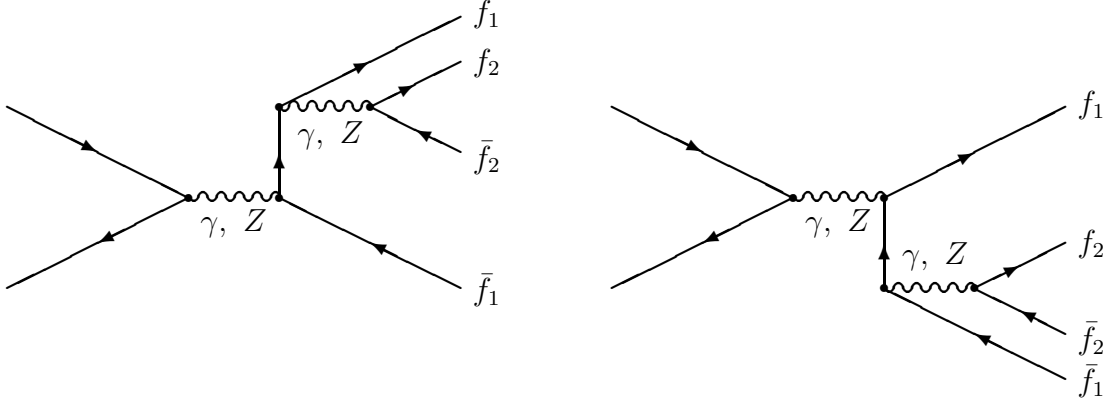
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Fig. 2: *The f_1-`deer` diagrams. The f_2-`deers` may be obtained by interchanging f_1 and f_2.*

between f_1-`deers` and f_2-`deers`, between `crabs` and f_2-`deers`, and between `crabs` and f_1-`deers`, respectively.

For the squares of diagrams of one group ($k = 1, 2, 3$), we obtain

$$\frac{\mathrm{d}^3\sigma_1(c, s, s_1, s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2} = \frac{\sqrt{\lambda}}{\pi s^2}C_{422}(e, s; f_1, s_1; f_2, s_2)\mathcal{T}_1\mathcal{G}^c_{422}(c; s; s_1, s_2)\,,$$

$$\frac{\mathrm{d}^3\sigma_2(c, s, s_1, s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2} = \frac{\sqrt{\lambda}}{\pi s^2}C_{422}(f_1, s_1; f_2, s_2; e, s)\mathcal{T}_2\mathcal{G}^c_{422}(c; s_1; s_2, s)\,,$$

$$\frac{\mathrm{d}^3\sigma_3(c, s, s_1, s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2} = \frac{\sqrt{\lambda}}{\pi s^2}C_{422}(f_2, s_2; e, s; f_1, s_1)\mathcal{T}_3\mathcal{G}^c_{422}(c; s_2; s, s_1)\,. \tag{12}$$

The color factor is $\mathcal{T}_k = N_c(f_1)N_c(f_2)$ for all $k = 1, \ldots, 6$, if photons or Z-bosons are exchanged. $N_c(f)$ is equal to unity for leptons and three for quarks. In the case of two different quarks in the final state, also gluon exchange is possible. The interference between diagrams with one gluon and those without a gluon gives $\mathcal{T}_k = 0$, the interference between two diagrams with a gluon gives $\mathcal{T}_k = 2/9$, $k = 2, 3, 4$. The corresponding contributions have to be added to σ_2, σ_3 and σ_4. The kinematical function $\mathcal{G}^c_{422}(c; s; s_1, s_2)$ is given in the Appendix. The function C_{422} contains couplings and gauge boson propagators,

$$\begin{aligned}
C_{422}(e, s; f_1, s_1; f_2, s_2) &= \frac{2s_1s_2}{(6\pi^2)^2}\,\Re e \sum_{V_i,V_j,V_k,V_l=\gamma,Z} \frac{1}{D_{V_i}(s_1)}\frac{1}{D_{V_j}(s_2)}\frac{1}{D^*_{V_k}(s_1)}\frac{1}{D^*_{V_l}(s_2)} \\
&\times [L(e, V_i)L(e, V_k)L(e, V_j)L(e, V_l) \\
&\qquad +R(e, V_i)R(e, V_k)R(e, V_j)R(e, V_l)] \\
&\times [L(f_1, V_i)L(f_1, V_k) + R(f_1, V_i)R(f_1, V_k)] \\
&\times [L(f_2, V_j)L(f_2, V_l) + R(f_2, V_j)R(f_2, V_l)]\,. \tag{13}
\end{aligned}$$

We use the following conventions for the left- and right-handed couplings between

vector bosons and a fermion f:

$$
\begin{aligned}
L(f,\gamma) &= R(f,\gamma) = \frac{eQ_f}{2}, \\
L(f,Z) &= \frac{e}{4s_W c_W}\left(2I_3^f - 2Q_f s_W^2\right), \quad R(f,Z) = \frac{e}{4s_W c_W}\left(-2Q_f s_W^2\right), \\
L(q,g) &= R(q,g) = \frac{1}{2}\sqrt{4\pi\alpha_s}.
\end{aligned}
\tag{14}
$$

The propagators are

$$
D_V(s) = s - M_V^2 + i\sqrt{s}\,\Gamma_V(s),
\tag{15}
$$

where M_V and $\Gamma_V(s) = \sqrt{s}\Gamma_V/M_V$ are the mass and width of the exchanged gauge boson ($M_\gamma = M_g = \Gamma_\gamma(s) = \Gamma_g(s) = 0$). Note that $\sigma_1(c,s,s_1,s_2)$ is of special interest because it describes the production and decay of two off-shell Z-bosons, see fig. 1. This is the numerically largest contribution, if the final fermions are not both quark pairs.

For the interferences between different groups of diagrams ($k = 4,5,6$), we obtain

$$
\begin{aligned}
\frac{d^3\sigma_4(c,s,s_1,s_2)}{dcds_1ds_2} &= \frac{\sqrt{\lambda}}{\pi s^2}C_{233}(e,s;f_1,s_1;f_2,s_2)\mathcal{T}_4\mathcal{G}_{233}^c(c;s;s_1;s_2)\ , \\
\frac{d^3\sigma_5(c,s,s_1,s_2)}{dcds_1ds_2} &= \frac{\sqrt{\lambda}}{\pi s^2}C_{233}(f_1,s_1;e,s;f_2,s_2)\mathcal{T}_6\mathcal{G}_{233}^c(c;s_1;s;s_2)\ , \\
\frac{d^3\sigma_6(c,s,s_1,s_2)}{dcds_1ds_2} &= \frac{\sqrt{\lambda}}{\pi s^2}C_{233}(f_2,s_2;e,s;f_1,s_1)\mathcal{T}_5\mathcal{G}_{233}^c(c;s_2;s;s_1)\ .
\end{aligned}
\tag{16}
$$

Again, the kinematical function $\mathcal{G}_{233}^c(c;s_2;s,s_1)$ is given in the Appendix. For C_{233} we get

$$
\begin{aligned}
C_{233}(e,s;f_1,s_1;f_2,s_2) &= \frac{2ss_1s_2}{(6\pi^2)^2}\,\Re e \sum_{V_i,V_j,V_k,V_l=\gamma,Z} \frac{1}{D_{V_i}(s)}\frac{1}{D_{V_j}(s_2)}\frac{1}{D_{V_k}^*(s)}\frac{1}{D_{V_l}^*(s_1)} \\
&\times\ [L(e,V_i)L(e,V_k) + R(e,V_i)R(e,V_k)] \\
&\times\ [L(f_1,V_i)L(f_1,V_j)L(f_1,V_l) - R(f_1,V_i)R(f_1,V_j)R(f_1,V_l)] \\
&\times\ [L(f_2,V_j)L(f_2,V_k)L(f_2,V_l) - R(f_2,V_j)R(f_2,V_k)R(f_2,V_l)]\ .
\end{aligned}
\tag{17}
$$

The triple differential distributions are described by only one combination of couplings for every interference, exactly as the double differential case [3]. However, the kinematical $\mathcal{G}$-functions are different. They depend on one more variable c which singles out the electron-positron pair and makes the $\mathcal{G}$-functions less symmetric.

We are now prepared to write the formulae for various distributions,

$$
\frac{d\sigma}{dc} = \int_{s_1^-}^{s_1^+} ds_1 \int_{s_2^-(s_1)}^{s_2^+(s_1)} ds_2 \frac{d^3\sigma(c,s,s_1,s_2)}{dcds_1ds_2},
$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y_1} = \int_{p_{1T}^-(y_1)}^{p_{1T}^+(y_1)} \mathrm{d}p_{1T} \int_{s_2^-(p_{1T},y_1)}^{s_2^+(p_{1T},y_1)} \mathrm{d}s_2 \frac{\mathrm{d}^3\sigma(c,s,s_1,s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2},$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}p_{1T}} = \int_{y_1^-(p_{1T})}^{y_1^+(p_{1T})} \mathrm{d}y_1 \int_{s_2^-(p_{1T},y_1)}^{s_2^+(p_{1T},y_1)} \mathrm{d}s_2 \frac{\mathrm{d}^3\sigma(c,s,s_1,s_2)}{\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2}. \tag{18}$$

The kinematical borders of integration and the transformations from c and s_1 to y_1 and p_{1T} are given in (7) - (10).

The distributions $\mathrm{d}^3\sigma(c,s,s_1,s_2)/(\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2)$ are derived for massless fermions. Hence, the formulae are not applicable in the phase space regions $s_1 \approx (m_1 + m_2)^2$, $s_2 \approx (m_3 + m_4)^2$. However, the mass effects can be completely removed by a moderate cut on s_1 and s_2. Such a cut should be applied to quark pairs in any case to remove non-perturbative QCD effects. To leading order, mass effects can be taken into account as in ref. [4].

The numerically most important radiative corrections to the considered process are due to initial state radiation of photons. They may be taken into account with a convolution formula which is well known from the Z line shape [2,6].

4. Summary

In this paper, compact analytical formulae for triple differential cross sections $\mathrm{d}^3\sigma(c,s,s_1,s_2)/(\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2)$ of the process (1) are presented. They make the semianalytic approach more flexible allowing the calculation of many distributions by at most two numerical integrations. Further, it allows the inclusion of cuts on the integration variables as summarized in the table:

observable	cuts possible on	No. of num. integrations
σ	s_1, s_2	2
$\mathrm{d}\sigma/\mathrm{d}s_1$	s_2	1
$\mathrm{d}\sigma/\mathrm{d}c$	s_1, s_2	2
$\mathrm{d}\sigma/\mathrm{d}p_{1T}$	y_1, s_2	2
$\mathrm{d}\sigma/\mathrm{d}y_1$	p_{1T}, s_2	2

Table 1: *Examples of cross sections and distributions calculable by the semianalytic approach.*

In addition to the observables indicated in the table, double differential distributions can also be calculated by less numerical integrations. The triple differential distribution $\mathrm{d}^3\sigma/(\mathrm{d}c\mathrm{d}s_1\mathrm{d}s_2)$ and the double differential distribution $\mathrm{d}^2\sigma/(\mathrm{d}s_1\mathrm{d}s_2)$ can be used as input for further investigations.

The formulae presented here, can easily be generalized to the exchange of extra heavy neutral gauge bosons by an extension of the summation in C_{422} and C_{233}. The generalization to longitudinal polarized beams is straightforward.

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Appendix A Formulae for the kinematical functions

We start with the kinematical function for `crab` diagrams squared, see fig. 1:

$$
\mathcal{G}_{422}^{CC}(c;s;s_1,s_2) = \frac{-1}{4}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)\frac{s^2+(s_1+s_2)^2}{s-s_1-s_2} - \frac{1}{4}\left(\frac{1}{T_1^2}+\frac{1}{T_2^2}\right)s_1 s_2 - \frac{1}{2}, \quad \text{(A.1)}
$$

with

$$
T_1 = -\frac{1}{2}\left(s-s_1-s_2+c\sqrt{\lambda}\right), \quad T_2 = -\frac{1}{2}\left(s-s_1-s_2-c\sqrt{\lambda}\right). \quad \text{(A.2)}
$$

The f_1-`deer` diagrams squared lead to the function

$$
\begin{aligned}
\mathcal{G}_{422}^{DD}(c;s_1;s_2,s) &= \frac{3}{8}(1+c^2)\mathcal{G}_{422}(s_1;s_2,s) \\
&\quad + \frac{1-3c^2}{\lambda}s_1(s+s_2)\frac{3}{4}\left(1-2\mathcal{L}(s_1;s_2,s)\frac{ss_2}{s_1-s_2-s}\right). \text{(A.3)}
\end{aligned}
$$

The logarithm $\mathcal{L}(s;s_1,s_2)$ is defined as

$$
\mathcal{L}(s;s_1,s_2) = \frac{1}{\sqrt{\lambda}}\ln\frac{s-s_1-s_2+\sqrt{\lambda}}{s-s_1-s_2-\sqrt{\lambda}}. \quad \text{(A.4)}
$$

$\mathcal{G}_{422}(s_1;s_2,s)$ is the corresponding kinematical function for the double differential distribution $\frac{d^2\sigma}{ds_1 ds_2}$ which we give here for sake of completeness [3]:

$$
\mathcal{G}_{422}(s;s_1,s_2) = \frac{s^2+(s_1+s_2)^2}{s-s_1-s_2}\mathcal{L}(s;s_1,s_2) - 2 \quad \text{(A.5)}
$$

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The function of f_2-deers squared is given by $\mathcal{G}_{422}^{DD}(c; s_2; s_1, s)$ which expresses the symmetry of the problem according to the two final fermion pairs.

We can unify the two functions (A.1) and (A.3) in one function $\mathcal{G}_{422}^{c}(c; s_1; s_2, s)$ to be close to the old notation used for double differential distributions:

$$\mathcal{G}_{422}^{c}(c; s; s_1, s_2) = \begin{cases} \mathcal{G}_{422}^{CC}(c; s; s_1, s_2) & \text{if } s > s_1 \text{ and } s > s_2 \\ \mathcal{G}_{422}^{DD}(c; s; s_1, s_2) & \text{else.} \end{cases} \tag{A.6}$$

As in the double differential case, $\mathcal{G}_{422}^{c}(c; s; s_1, s_2)$ is symmetric in the last two arguments.

The kinematical function of the interference between f_1-deers and f_2-deers is

$$\begin{aligned}
\mathcal{G}_{233}^{DD}(c; s; s_1, s_2) &= \frac{3}{8}(1 + c^2)\mathcal{G}_{233}(s; s_1, s_2) \\
&\quad - \frac{3}{\lambda^2}\frac{3}{8}(1 - 3c^2)s\Big[\mathcal{L}(s_1; s_2, s)2s_2(s_1 - s_2) + (s - s_1 - 3s_2)\Big] \\
&\quad \times \Big[\mathcal{L}(s_2; s, s_1)2s_1(s_2 - s_1) + (s - s_2 - 3s_1)\Big]. \tag{A.7}
\end{aligned}$$

$\mathcal{G}_{233}(s; s_1, s_2)$ is the kinematical function for the double differential distribution [3]:

$$\begin{aligned}
\mathcal{G}_{233}(s; s_1, s_2) &= \frac{3}{\lambda^2}\Big\{\mathcal{L}(s_2; s, s_1)\mathcal{L}(s_1; s_2, s) \\
&\quad 4s\Big[ss_1(s - s_1)^2 + ss_2(s - s_2)^2 + s_1 s_2(s_1 - s_2)^2\Big] \\
&\quad + (s + s_1 + s_2)\Big[\mathcal{L}(s_2; s, s_1)2s\Big[(s - s_2)^2 + s_1(s + s_2 - 2s_1)\Big] \\
&\quad + \mathcal{L}(s_1; s_2, s)2s\Big[(s - s_1)^2 + s_2(s + s_1 - 2s_2)\Big] \\
&\quad + 5s^2 - 4s(s_1 + s_2) - (s_1 - s_2)^2\Big]\Big\}. \tag{A.8}
\end{aligned}$$

The interference between crabs and f_1-deers is given by:

$$\begin{aligned}
\mathcal{G}_{233}^{CD}(c; s_2; s; s_1) &= \frac{3}{\lambda^2}\Big\{\frac{-1}{4}\Big(\frac{1}{T_1} + \frac{1}{T_2}\Big)\mathcal{L}(s_1; s_2, s) \\
&\quad \times 4s_2\Big[ss_1(s - s_1)^2 + ss_2(s - s_2)^2 + s_1 s_2(s_1 - s_2)^2\Big] \\
&\quad + (s + s_1 + s_2)\Big[\frac{-1}{4}\Big(\frac{1}{T_1} + \frac{1}{T_2}\Big)2s_2\Big[(s - s_2)^2 + s_1(s + s_2 - 2s_1)\Big] \\
&\quad + \mathcal{L}(s_1; s_2, s)s_2\Big[(s_1 - s_2)^2 + s(s_1 + s_2 - 2s)\Big] \\
&\quad + \frac{1}{2}[5s_2^2 - 4s_2(s + s_1) - (s - s_1)^2]\Big]\Big\}. \tag{A.9}
\end{aligned}$$

The interference between `crabs` and f_2-`deers` is expressed by $\mathcal{G}^{CD}_{233}(c; s_1; s; s_2)$. In contrast to the double differential case, $\mathcal{G}^{CD}_{233}(c; s_2; s; s_1)$ is not symmetric under the exchange of the last two arguments. Again we write the functions (A.7) and (A.9) formally as one:

$$\mathcal{G}^c_{233}(c; s; s_1; s_2) = \begin{cases} \mathcal{G}^{DD}_{233}(c; s; s_1, s_2) & \text{if } s > s_1 \text{ and } s > s_2 \\ \mathcal{G}^{CD}_{233}(c; s; s_1; s_2) & \text{else.} \end{cases} \quad (A.10)$$

All kinematical functions are finite and independent of c for $\lambda \to 0$ (unfortunately, in ref. [3] is a typing error in the formula for the limit of $\mathcal{G}_{233}(s; s_1; s_2)$):

$$\mathcal{G}^c_{422}(c; s; s_1, s_2) \rightarrow \frac{s(s_1 + s_2)}{2 s_1 s_2}$$

$$\mathcal{G}^c_{233}(c; s; s_1; s_2) \rightarrow \frac{s_1 + s_2 - 5s}{(s_1 - s_2 - s)(s_2 - s - s_1)} \quad (A.11)$$

Finally, we remark that the above functions give those of the double differential distributions, if integrated over c:

$$\int_{-1}^{1} dc \, \mathcal{G}^c_{422}(c; s; s_1, s_2) = \mathcal{G}_{422}(s; s_1, s_2),$$

$$\int_{-1}^{1} dc \, \mathcal{G}^c_{233}(c; s; s_1; s_2) = \mathcal{G}_{233}(s; s_1, s_2). \quad (A.12)$$